Available online at www.sciencedirect.com
SCIENCE DIRECT

ELSEVIER

Journal of APPLIED MATHEMATICS AND MECHANICS

www.elsevier.com/locate/jappmathmech

Journal of Applied Mathematics and Mechanics 69 (2005) 720–725

THE CAUCHY–RIEMANN CONDITIONS AND LOCALIZED ASYMPTOTIC SOLUTIONS OF THE LINEARIZED SHALLOW-WATER EQUATIONS†

S. Yu. DOBROKHOTOV, B. TIROZZI and A. I. SHAFAREVICH

Moscow

email: shafar@mech.math.msu.su

(*Received* 30 *August* 2004)

Singular solutions of the two-dimensional shallow-water equations with algebraic singularities of the "square root" type, which have been studied before [1–4], propagate along the trajectories of the external velocity field, over which this field satisfies the Cauchy–Riemann conditions. In other words, the differential of the phase flow on such a trajectory is proportional to an orthogonal operator. It turns out that in the linear approximation this situation is strongly linked with the "spreading" effect of solutions of the hydrodynamic equations (cf. [5, 6]); namely, a localized asymptotic solution of the Cauchy problem for the linearized shallow-water equations maintains its form (i.e. does not spread) if and only if the Cauchy–Riemann conditions hold on the trajectory of the outer flow along which the disturbance is propagating.

1. THE CONSTRUCTION OF LOCALIZED SOLUTIONS OF LINEARIZED SHALLOW-WATER EQUATIONS

Let $V(x, t)$ be a smooth vector field in $\mathbb{R}^2$ which depends smoothly on the time t, and $\eta_0(x, t)$ a smooth scalar function; the shallow-water in the β-plane approximation, linearized over the velocity field V and geopotential η_0, are

$$\begin{aligned} &\frac{\partial u}{\partial t} + (V, \nabla)u + (u, \nabla)V + \Omega T u + \nabla\eta = 0, \quad \Omega = \Omega_0 + \beta x_2, \quad T = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \\ &\frac{\partial \eta}{\partial t} + (V, \nabla)\eta + (u, \nabla)\eta_0 + \eta(\nabla, V) + \eta_0(\nabla, u) = 0 \end{aligned} \tag{1.1}$$

where Ω is the Coriolis frequency in the β plane.

Let us consider a Cauchy problem for this system of equations, with an initial condition localized in a small neighbourhood of a point $x_0 \in \mathbb{R}^2$

$$t = 0: u = u^0\left(\frac{x - x_0}{h}\right), \quad \eta = \eta^0\left(\frac{x - x_0}{h}\right), \quad h \to 0 \tag{1.2}$$

where $u^0(y)$ and $\eta^0(y)$ are smooth functions that decrease sufficiently rapidly as $|y| \to \infty$ (to fix our ideas, we shall assume that they are Schwartzian) and h is a small parameter characterizing the "localization" of the initial data.

†*Prikl. Mat. Mekh.* Vol. 69, No. 5, pp. 804–809, 2005.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.09.006

The solution of problem (1.1), (1.2) will be sought in the form

$$u(x,t,h) = v\left(\frac{S(x,t)}{h}, x, t\right) + hv_1\left(\frac{S(x,t)}{h}, x, t\right) + \dots$$
$$\eta(x,t,h) = \rho\left(\frac{S(x,t)}{h}, x, t\right) + h\rho_1\left(\frac{S(x,t)}{h}, x, t\right) + \dots \qquad (1.3)$$

where $S(x, t) = (S_1, S_2)$ is a continuous two-dimensional vector-valued function, and the vector fields $v(y, x, t)$, $v_1(y, x, t)$ and the scalar functions $\rho(y, t)$, $\rho_1(y, t)$ depend continuously on their arguments and decrease as $|y| \to \infty$. The functions S_1 and S_2 describe the localization of the asymptotic solution: It is concentrated in a small neighbourhood of the set $S = 0$. This set is assumed to consist of a single point for each t, that is, the vector-valued function S is assumed to vanish at a single point $R(t) \in \mathbb{R}^2$, and moreover S_1 and S_2 define curvilinear coordinates in the neighbourhood of the point (this means that the vectors ∇S_1 and ∇S_2 are linearly independent).

Let us substitute the functions (1.3) into Eqs (1.1) and successively equate the coefficients of all powers of the small parameter h to zero. Equating the coefficient of h^{-1} to zero, we obtain the equations.

$$(\omega, \nabla_y)v + \frac{\partial S^*}{\partial x}\nabla_y\rho = 0, \quad (\omega, \nabla_y)\rho + \eta_0\left(\frac{\partial S^*}{\partial x}\nabla_y, v\right) = 0; \quad \omega = \frac{\partial S}{\partial t} + (V, \nabla)S \qquad (1.4)$$

where ∇_y is the gradient with respect to the "fast" variable $y = S/h$, $\partial S/\partial x$ is the 2×2 matrix of derivatives of S, and the asterisk denotes transposition. Taking the Fourier transform of system (1.4) with respect to the variable y, we obtain

$$(k, \omega)(p, \tilde{v}) + p\tilde{\rho} = 0, \quad (k, \omega)\tilde{\rho} + \eta_0(p, \tilde{v}) = 0 \qquad (1.5)$$

$$k = \begin{Vmatrix} k_1 \\ k_2 \end{Vmatrix}, \quad p = \frac{\partial S^*}{\partial x}k = k_1\nabla S_1 + k_2\nabla S_2$$

where k is the variable dual to y and the tilde denotes Fourier transforms with respect to y.

Lemma 1. System (1.5) in $\tilde{v}$ and $\tilde{\rho}$ has a non-trivial solution if $\omega = 0$. This solution is

$$\tilde{\rho} = 0, \quad \tilde{v} = n\theta$$

where $n = (-p_2, p_1)$ is a vector orthogonal to p, and θ is an arbitrary scalar function.

Proof. Multiplying the first (vector) equation of system (1.5) by the vectors p and n, we obtain a system of three equations

$$(k, \omega)(p, \tilde{v}) + p^2\tilde{\rho} = 0, \quad (k, \omega)\tilde{\rho} + \eta_0(p, \tilde{v}) = 0, \quad (k, \omega)(\tilde{v}, n) = 0 \qquad (1.6)$$

The first two equations form a linear homogeneous system in $\tilde{\rho}$ $(p, \tilde{v})$. This system has a non-trivial solution only if $(k, \omega)^2 = p^2\eta_0$, and this equality cannot hold under the assumptions formulated above concerning the vector-valued function S. Indeed, since the vectors ∇S_1 and ∇S_2 are linearly independent, the vector p does not vanish for $k \neq 0$, that is, the quadratic form in k equal to p^2 is positive-definite. On the other hand, the quadratic form $(k, \omega)^2$ always has a kernel – the set of vectors orthogonal to ω. Thus, the first two equations of system (1.6) have only trivial solutions. A non-trivial solution of the last (third) equation exists if $(k, \omega) = 0$; this is true for all k if and only if $\omega = 0$. In that case the projection of the vector $\tilde{v}$ onto the direction of n is arbitrary.

Corollary 1. The solution of the Cauchy problem (1.1), (1.2) is of the form (1.3) only if the initial data satisfy the relations

$$\eta^0 = 0, \quad (k, \tilde{u}^0) = 0 \qquad (1.7)$$

This last condition means that the initial field u^0 is non-divergent, that is, it belongs to the so-called hydrodynamic mode (see also the remark below).

In what follows we shall assume that the initial data satisfy conditions (1.7); note that in that case

$$\tilde{u}^0 = n_0\theta_0, \quad n_0 = \left\| \begin{matrix} -k_2 \\ k_1 \end{matrix} \right\|$$

where $\theta_0(k)$ is a scalar function.

Corollary 2. The vector-valued function $S(x, t)$ satisfies the linear equation

$$\frac{\partial S}{\partial t} + (V, \nabla)S = 0 \tag{1.8}$$

Comparing the form of the asymptotic solution (1.3) with the initial condition (1.2), we obtain the initial condition

$$t = 0: S = x - x_0 \tag{1.9}$$

The solution of the Cauchy problem (1.8), (1.9) is

$$S(x, t) = g_t^{-1}x - x_0$$

where g_t is the phase flow of the field V. In other words, $S(x, t) + x_0$ is the initial point of a trajectory of the field that arrives in time t at the point x.

Remark. The characteristics of the shallow-water equations may be divided into two types: hydrodynamic and acoustic. The equation of the hydrodynamic characteristics has the form (1.8); the characteristics themselves are the trajectories of the field V. The acoustic characteristics are described by the eikonal equation

$$\left(\frac{\partial s}{\partial t} + (V, \nabla s)\right)^2 = \eta_0^2(\nabla s)^2$$

Lemma 1 implies that localized solutions of type (1.3) only propagate along hydrodynamic characteristics, that is, no acoustic modes of this type appear.

The typical behaviour of acoustic modes with localized initial conditions for arbitrary hyperbolic equations was described in [7]. As a rule, after an arbitrarily small time, such solutions become functions localized near a set of codimension 1 (in the case under consideration – near a curve in plane). Thus, conditions (1.7), which single out the hydrodynamic mode, thereby guarantee localization of the solution near the moving point.

Now consider the terms of order h^0 that arise when the functions (1.3) are substituted into Eqs (1.1). Equating such terms in the first (vector) equation to zero and taking Fourier transformations with respect to y, we obtain

$$\dot{\tilde{v}} + \frac{\partial V}{\partial x}\tilde{v} + \Omega T\tilde{v} + ip\tilde{\rho}_1 = 0 \tag{1.10}$$

where the dot denotes differentiation along trajectories of the vector field V. It is clear that the initial condition for the function $\tilde{v}$ has the form

$$t = 0: \tilde{v} = \tilde{u}^0 = n_0\theta_0(k)$$

Lemma 2. The function $\tilde{v}$ has the form

$$\tilde{v} = n(k)\frac{k^2}{p^2}\theta_0(k)\exp\left(-\int_0^t(\nabla, V)(z, t_1)dt_1\right)\Bigg|_{z = g_t^{-1}x}$$

Proof. Lemma 1 states, in particular, that $\tilde{v} = n\theta$; we substitute this form of the solution into Eq. (1.1). Multiplying the resulting equality by the vector n, we find that

$$n^2\dot{\theta} + (n, \dot{n})\theta + \left(n, \frac{\partial V}{\partial x}n\right)\theta = 0 \tag{1.11}$$

Differentiating Eq. (1.8), we obtain

$$\dot{p} = -\frac{\partial V^*}{\partial x}p$$

Multiplying this equation by the matrix T of rotation by $\pi/2$ and taking into account the following equality, which is true for any 2×2 matrix A,

$$-TA^*T^{-1} = A - \operatorname{tr}A$$

we find that

$$\dot{n} = -T\frac{\partial V^*}{\partial x}T^{-1}n = \frac{\partial V}{\partial x}n - (\nabla, V)n$$

Substituting the expression for $V_x n$ resulting from this equality into Eq. (1.11) and using the fact that $n^2 = p^2$, we get

$$\left(\frac{d}{dt} + (\nabla, V)\right)(p^2\theta) = 0$$

whence the required formula at once follows (d/dt denotes differentiation along a trajectory of V).

We can now formulate our main result.

Theorem. The solution of the Cauchy problem (1.1), (1.2) has the form

$$\begin{aligned} u(x,t,h) &= \frac{E(t)}{2\pi}\int_{\mathbf{R}^2} n(p)\frac{k^2}{p^2}\theta_0(k)\exp\left(i\frac{(k,S)}{h}\right)dk\Big|_{S = g_t^{-1}x - x_0} + w \\ E(t) &= \exp\left(-\int_0^t(\nabla, V)dt_1\right) \end{aligned} \tag{1.12}$$

where $|w| \to 0$ as $h \to 0$.

The proof amounts to computing a few corrections to the leading part of the asymptotic expansion described above and subsequently estimating the remainder. To obtain such an estimate one can, for example, use the representation of the resolvent operator of the Cauchy problem for Eqs (1.1) as an asymptotic series with respect to smoothness (cf. [8]).

2. SPREADING OF THE LOCALIZED SOLUTION AND THE CAUCHY-RIEMANN CONDITIONS

Note that the integrand in formula (1.12) need not be smooth at the point $k = 0$, since the quotient k^2/p^2 is not continuous there. As a result, the function $\upsilon(y,x,t)$ decreases as $|y| \to \infty$ like $O(|y|^2)$, that is, the initial condition "spreads". If further conditions are imposed on the field V, however, this spreading may disappear.

Proposition. As $h \to 0$, the leading part of the integrand in (1.12) is smooth if and only if the Cauchy–Riemann conditions hold on a trajectory $x = g_t x_0$ of the field V emanating from the point x_0:

$$\frac{\partial V_1}{\partial x_1} = \frac{\partial V_2}{\partial x_2}, \quad \frac{\partial V_1}{\partial x_2} = -\frac{\partial V_2}{\partial x_1}.$$

In that case the leading part of the asymptotic solution of problem (1.1), (1.2) is

$$U = \sqrt{E(t)}R(t)u^0\left(\frac{g_t^{-1}x - x_0}{h}\right) \tag{2.1}$$

where

$$R(t) = \begin{Vmatrix} \cos\varphi(t) & \sin\varphi(t) \\ -\sin\varphi(t) & \cos\varphi(t) \end{Vmatrix}, \quad \varphi(t) = \int_0^t \frac{\partial V_1}{\partial x_2} dt = \frac{1}{2}\int_0^t \operatorname{rot} V dt$$

that is, the solution exactly duplicates the form of the initial disturbance.

Proof. We note first of all that, since $\upsilon(y, x, t) = O(|y|^{-2})$, the "slow" variable x in this function may be replaced mode $o(1)$ by its value $g_t x_0$ on a trajectory emanating from the point x_0; this follows from the fact that

$$|x - g_t x_0| = |g_t(S + x_0) - g_t x_0| = O(|S|) = O(h|y|) \quad \text{as} \quad S \to 0$$

It is furthermore clear that the integrand in (1.12) is smooth if an only if $p^2 = k^2\lambda$, where λ is independent of k. Since the vector p satisfies the equations

$$\dot{p} = -\frac{\partial V^*}{\partial x} p, \quad p(0) = k$$

this condition means that the Cauchy operator of the system differs by a scalar factor from an orthogonal operator, i.e. the matrix $(\partial V/\partial x)^*$, and therefore also $\partial V/\partial x$, differ from a skew-symmetric operator by a scalar term. In that case, therefore, we have

$$\partial V/\partial x = \mu I + \nu T$$

where I is the 2×2 identity matrix, whence it follows that

$$\frac{\partial V_1}{\partial x_1} = \frac{\partial V_2}{\partial x_2} = \mu, \quad \frac{\partial V_1}{\partial x_2} = -\frac{\partial V_2}{\partial x_1} = \nu$$

that is, the Cauchy–Riemann conditions hold on the trajectory $x = g_t x_0$. Suppose these equalities indeed hold. Then

$$p^2 = \exp\left(-2\int_0^t \mu dt\right) k^2 = E(t) k^2, \quad n(p) = \sqrt{E}(t) R(t) n(k)$$

whence expression (2.1) follows at once.

Example. Evolution of a packet with exponential profile. As a simple example, let us consider an initial velocity field

$$u^0(y) = \mathfrak{D} \exp\left(-\frac{y^2}{2}\right), \quad \mathfrak{D} = \begin{Vmatrix} -\partial/\partial y_2 \\ \partial/\partial y_1 \end{Vmatrix}$$

If the Cauchy–Riemann condition holds on a trajectory of the outer flow, then the leading part of the asymptotic expansion of the solution of problem (1.1), (1.2) is

$$U = \sqrt{E(t)} R(t) \exp\left(-\frac{y^2}{2}\right)\Bigg|_{y = (g_t^{-1} x - x_0)/h}$$

But if the condition fails to hold, then

$$U = \frac{iE^2}{2\pi} A(t) \mathfrak{D} \int_0^{2\pi} \frac{k^2}{p^2}(\varphi) \exp\left(-\frac{\lambda^2}{4h^2}\right) D_{-2}\left(\frac{-i\lambda}{h}\right) d\varphi$$

$$\lambda(x, t, \varphi) = (e(\varphi), x - g_t x_0), \quad e(\varphi) = (\cos\varphi, \sin\varphi)$$

$$D_{-2}(t) = \exp\left(-\frac{t^2}{4}\right) \int_0^\infty \exp\left(-tr - \frac{r^2}{2}\right) r dr$$

where φ is the polar angle in the plane of the variables k, D_{-2} is the function of a parabolic cylinder of order -2, and A is the Cauchy operator of the system

$$\dot{z} = \frac{\partial V}{\partial x} z$$

This research was supported financially by the Russian Foundation for Basic Research (02-01-00850).

REFERENCES

1. MASLOV, V. P., Three algebras corresponding to non-smooth solutions of quasi-linear hyperbolic equations. *Uspekhi Mat. Nauk*, 1980, **35**, 2, 252–253.
2. DOBROKHOTOV, S. Yu., Hugoniot-Maslov chains for solitary vortices of the shallow water equations, Pt 1. *Rus. J. Math. Physics*, 1999, **6**, 2, 137–173; Pt 2, ibid., 1999, **6**, 3, 282–313.
3. DOBROKHOTOV, S. Yu. and TIROZZI, B., The Hamiltonian property of the truncated Hugoniot-Maslov chain for trajectories of mesoscale vortices. *Dokl. Ross. Akad. Nauk*, 2002, **384**, 6, 741–746.
4. DOBROKHOTOV, S. Yu., SEMENOV, Ye. V. and TIROZZI, B., Hugoniot-Maslov chains for solitary vortical solutions of quasi-linear hyperbolic systems and typhoon trajectories. *Sovremennye Problemy Matematiki. Fundamental'nye Napravleniya*, 2003, **2**, 5–44.
5. DOBROKHOTOV, S. Yu. and SHAFAREVICH, A. I., The parametrix and asymptotic form of localized solutions of the Navier–Stokes equations in R^3, linearized on a smooth flow. *Mat. Zametki*, 1992, **51**, 1, 72–82.
6. DOBROKHOTOV, S. Yu. and SHAFAREVICH, A. I., The behaviour at infinity of the velocity field of an incompressible fluid. *Izv. Ros. Akad. Nauk. MZhG*, 1996, 4, 38–42.
7. DOBROKHOTOV, S. Yu., ZHEVANDROV, P. N., MASLOV, V. P. and SHAFAREVICH, A. I., Asymptotic rapidly decreasing solutions of linear strongly hyperbolic systems with variable coefficients. *Mat. Zametki*, 1991, **49**, 4, 31–46.
8. MASLOV, V. P., *Asymptotic Methods of Solving Pseudodifferential Equations*. Nauka, Moscow, 1987.

Translated by D.L.